Optimization and Numerical Linear Algebra Past Quals Solutions

Trevor Loe

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This document contains solutions that I came up with for past ONLA qual problems. This is in no way a complete solution guide to all the past quals, but I have tried to cover as many past problems as possible.

Thank you for Zerrin Vural,

Past ONLA qualifying exams can be found at `https://ww3.math.ucla.edu/past-qualifying-exams/`

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1a

Find the critical points and local extremizers of

$$f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3$$

subject to

$$x_1^2 + x_2^2 + x_3^2 = 4$$

To solve this, we use Lagrange's theorem, stating that if x^* is a local minizer of f subject to $h(x) = 0$ then (as long as $\nabla h_1, ..., \nabla h_m$ are linearly independent, there exists some λ such that

$$Df(x^*) + \lambda^T Dh(x^*) = 0$$

We have

$$Df(x) = \begin{pmatrix} 2x_1 \\ 6x_2 \\ 1 \end{pmatrix}$$

and as there is only one h function, we just have

$$Dh(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}$$

so we want a λ and x_1, x_2, x_3 solving

$$\begin{pmatrix} 2x_1 \\ 6x_2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} = 0 \quad x_1^2 + x_2^2 + x_3^2 = 4$$

From the third equation we have $x_3 = -1/(2\lambda)$. From the first equation we have

$$x_1(1 + \lambda) = 0$$

So either $x_1 = 0$ or $\lambda = -1$. First consider when $x_1 = 0$. We then have

$$x_2^2 + x_3^2 = 4$$

and

$$x_2(3 + \lambda) = 0$$

So $x_2 = 0$ or $\lambda = -3$. First consider when $x_2 = 0$. Then $x_3 = \pm 2$ and $\lambda = \pm(-1/4)$. So two solutions are

$$(x_1, x_2, x_3, \lambda) = (0, 0, 2, -1/4), (0, 0, -2, 1/4)$$

Next we consider when $\lambda = -3$. Then $x_3 = 1/6$. Thus

$$x_2^2 + 1/36 = 4$$

Meaning $x_2 = \pm\sqrt{\frac{143}{36}}$. So two more solutions are

$$(0, \sqrt{\frac{143}{36}}, 1/6, -3), (0, -\sqrt{\frac{143}{36}}, 1/6, -3)$$

Now we consider the case where $x_1 \neq 0$, so $\lambda = -1$. For this case we have $x_3 = 1/2$. Also $6x_2 - 2x_2 = 0$ so $x_2(6-2) = 0$ so $x_2 = 0$. So we get

$$x_1^2 + 1/4 = 4$$

meaning $x_1 = \pm\sqrt{\frac{15}{16}}$. So our final two solutions are

$$(\sqrt{\frac{15}{16}}, 0, 1/2, -1), (-\sqrt{\frac{15}{16}}, 0, 1/2, -1)$$

We can summarize our results as follows To figure out if these points are minimizers for the function, we need

x_1	x_2	x_3	λ
0	0	2	$-\frac{1}{4}$
0	0	-2	$\frac{1}{4}$
0	$\sqrt{\frac{143}{36}}$	$\frac{1}{6}$	-3
0	$-\sqrt{\frac{143}{36}}$	$\frac{1}{6}$	-3
$\sqrt{\frac{15}{16}}$	0	$\frac{1}{2}$	-1
$-\sqrt{\frac{15}{16}}$	0	$\frac{1}{2}$	-1

Table 1: Stationary points to function

the second order necessary/sufficient conditions, which would require that

$$L(x, \lambda) = F(x) + \lambda H(x)$$

is positive semi-definite or positive definite (but only for points in the tangent space). Similar, for negative semi-definiteness, we would get a local max rather than a min.
We can compute L as follows

$$L(x, \lambda) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

If $\lambda = -1/4$ we get

$$L = \begin{pmatrix} 3/2 & 0 & 0 \\ 0 & 11/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

We also have that the tangent space at the first point is

$$T(x^*) = \{y \in \mathbb{R}^3 : \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} y = 0\} = \{\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R}\}$$

Note that, for any one of these vectors we get that $y^T Ly > 0$. Meaning that this point satisfies the second order sufficient condition and is a minimizer.
Next, note that the second point has

$$\begin{pmatrix} 5/2 & 0 & 0 \\ 0 & 13/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

which is pos. definite for any $y \in \mathbb{R}^3$, so in particular, for any $y \in T(x^*)$ we get $y^T Ly > 0$. So this point is also a minimizer.
Next, when $\lambda = -3$ we get

$$L = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

and for this one we have

$$T(x^*) = \{y \in \mathbb{R}^3 : \begin{pmatrix} 0 \\ \sqrt{143}/3 \\ 1/3 \end{pmatrix} y = 0\}$$

This point will satisfy the necessary condition and the sufficient condition, because the only vector that wouldn't be negative from L is $(0, a, 0)^T$. So this point is local maximizer. Similarly, for the third point we get an identical tangent space except with $-\sqrt{143}$. So that will also be a local maximizer.
Now for the fifth and sixth point, we have $\lambda = -1$ so

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

For the fifith point we get a tangent space like

$$y = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $a\sqrt{\frac{15}{16}} + c/2 = 0$. As b can be anything, we have $(0, 1, 0) \in T(x^*)$ which would give us a positive number from the quadratic form, but $(a, 0, c)$ would give us a negative. So the matrix is indefinite, meaning the necessary conditions are not satisfied.

1b

Find the solutions to

$$\max x^T \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix} x$$

subject to

$$||x|| = 1$$

Similarly, we will use the Lagrange conditions to find the stationary points. We have

$$Df(x) = A^T x + Ax = (A^T + A)x = \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} x = \begin{pmatrix} 6x_1 + 5x_2 \\ 5x_1 + 6x_2 \end{pmatrix}$$

Similarly,

$$Dh(x) = D(\sqrt{x_1^2 + x_2^2}) = \begin{pmatrix} \frac{x_1}{||x||} \\ \frac{x_2}{||x||} \end{pmatrix}$$

So the Lagrange conditions are

$$\begin{pmatrix} 6x_1 + 5x_2 \\ 5x_1 + 6x_2 \end{pmatrix} + \lambda \begin{pmatrix} \frac{x_1}{||x||} \\ \frac{x_2}{||x||} \end{pmatrix} = 0$$

and

$$||x|| = 1$$

Note that the second equation implies that the first two equations must satisfy

$$6x_1 + 5x_2 + \lambda x_1 = 0 \quad 5x_1 + 6x_2 + \lambda x_2 = 0$$

and the last equation implies that

$$x_1^2 + x_2^2 = 1$$

We then can solve

$$x_1(6 + \lambda) = -5x_2$$

and then

$$x_2^2 \left(\left(\frac{5}{6+\lambda} \right)^2 + 1 \right) = 1$$

Also subbing into the second equation we get

$$5 \frac{-5}{6+\lambda} x_2 + (6+\lambda)x_2 = 0$$

So either $x_2 = 0$ or

$$6 + \lambda - \frac{25}{6+\lambda} = 0$$

But if $x_2 = 0$ then $x_1 = 0$ which is clearly not a solution. So we have

$$(6+\lambda)^2 - 25 = 0$$

and thus

$$36 + 12\lambda + \lambda^2 - 25 = \lambda^2 + 12\lambda + 11 = 0$$

which means $\lambda = -11$ or $\lambda = -1$. If $\lambda = -11$, then

$$x_2^2((5/5)^2 + 1) = 1$$

so $x_2 = \sqrt{1/2}$. Consequently $x_1 = x_2 = \sqrt{1/2}$. So our first solution is

$$(x_1, x_2, \lambda) = (1/\sqrt{2}, 1/\sqrt{2}, -11)$$

Now consider when $\lambda = -1$. In this case,

$$x_2^2(2) = 1$$

so $x_2 = 1/\sqrt{2}$. Also $x_1(5) = -5x_2$. So $x_1 = -x_2 = -1/\sqrt{2}$. Thus, our other solution is

$$(-1/\sqrt{2}, 1/\sqrt{2}, -1)$$

We can now check the second order conditions by considering first that $||x|| = 1$ if and only if $||x||^2 = 1$. So $h(x) = x_1^2 + x_2^2$ and we get

$$F(x) = \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}$$

$$H(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So

$$L(x, \lambda) = \begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

If $\lambda = -11$ we get

$$L(x, \lambda) = \begin{pmatrix} -16 & 5 \\ 5 & -16 \end{pmatrix}$$

This matrix has eigenvalues -21 and -11, so it is negative definite. Thus the first point is a maximizer. We would expect that, being a quadratic form, the function has a unique max. But just for fun, we can compute L for $\lambda = -1$ to be

$$L(x, \lambda) = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$$

Which has eigenvalues 9 and -1. The tangent space for this point will be vectors of the form $(1, 1)$, which just so happens to be the eigenvector for the 9 eigenvalue. So this second point is a minimizer.

2

Let

$$B = \begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$$

With A square the $||A||_2 \leq 1$. Prove that the condition number of B satisfies

$$\kappa(B) = \frac{1 + ||A||_2}{1 - ||A||_2}$$

For this problem we first consider the SVD of A given by

$$A = U\Sigma V^*$$

Then note we can write B as

$$B = \begin{pmatrix} I & U\Sigma V^* \\ V\Sigma U^* & I \end{pmatrix}$$

Now consider the vector $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$, where $Av_1 = \sigma_{max} u_1$, the largest singular value for A. We then get

$$B \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} I & U\Sigma V^* \\ V\Sigma U^* & I \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_1 + \sigma_{max} u_1 \\ v_1 \sigma_{max} + v_1 \end{pmatrix} = (1 + \sigma_{max}) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

Note that for any vector of that form, the upper vector cannot grow in magnitude by more than $1 + \sigma_{max}$ as it will be the identity matrix plug a vector scaled by A.

Similarly, note that if we multiply by $\begin{pmatrix} u_1 \\ -v_1 \end{pmatrix}$ we get

$$B \begin{pmatrix} u_1 \\ -v_1 \end{pmatrix} = (1 - \sigma_{max}) \begin{pmatrix} u_1 \\ -v_1 \end{pmatrix}$$

Observe that this is smallest scaling that can be applied to the vector for the same reason as above. Thus, we have found the maximum scaling that B can achieve, which is in fact an eigenvalue for B, and the smallest. Thus we get

$$\overline{\sigma_{max}} = 1 + \sigma_{max}$$

where $\overline{\sigma_{max}}$ denotes the largest singular value of B. Similarly,

$$\overline{\sigma_{min}} = 1 - \sigma_{max}$$

and σ_{max} denotes the largest singular value for A. Thus, the condition number for B, being the ratio of the largest to the smallest singular value is

$$\kappa(B) = \frac{1 + \sigma_{max}}{1 - \sigma_{max}}$$

3

The Jacobi iteration to solve $Ax = b$ is given by

$$x^{(k+1)} = M^{-1} N x^{(k)} + M^{-1} b$$

where

$$M = D \quad N = -(L + U)$$

And the Jacobi Over-Relaxation is

$$M = \frac{1}{\omega} D \quad N = -\left(\left(1 - \frac{1}{\omega}\right) D + L + U\right)$$

Prove that if Jacobi converges, then the over-relaxation also converges, when $\omega \in (0, 1]$.
To prove this, suppose that Jacobi converges. Suppose that x^* is the solution we seek so

$$Ax^* = b$$

Note that

$$x^{(k+1)} - x^* = M^{-1}Nx^{(k)} + M^{-1}b - x^* = (I - M^{-1}A)x^{(k)} + M^{-1}Ax^* - x^*$$

and we can factor out to the form

$$x^{(k+1)} - x^* = (I - M^{-1}A)(x^{(k)} - x^*)$$

Let $G = I - M^{-1}A$. Then we have that the method will converg if and only if $||G|| < 1$ where the norm here denotes the spectral radius, the largest eigenvalue. We suppose that regular Jacobi converges, meaning that for G define in Jacobi, namely $||\tilde{G}|| < 1$. Also denote $\tilde{M} = D$ and $\tilde{N} = -(L + U)$. Then we get for the over-relation case

$$G = \left(\frac{1}{\omega}D\right)^{-1}\left(-\left(\left(1 - \frac{1}{\omega}\right)D + L + U\right)\right)$$

Note that, as D is diagonal, its inverse is just the matrix with the inverse of its elements. Namely,

$$\left(\frac{1}{\omega}D\right)^{-1} = \omega D^{-1}$$

So

$$G = -\omega D^{-1}\left(\left(1 - \frac{1}{\omega}\right)D + L + U\right) = (1 - \omega)I - \omega D^{-1}(L + U)$$

Now observe that $-D^{-1}(L + U) = \tilde{G}$. Now let u be an arbitrary unit vector. By the triangle inequality

$$||Gu|| \leq (1 - \omega)||u|| + \omega||\tilde{G}u|| < (1 + \omega) + \omega = 1$$

As the spectral radius is bounded by the operator norm we get that the spectral radius must then be less than 1. So the over-relaxation method will converge.

4

a

The Lanczos iteration will tridiagonalize a hermitian matrix A to the form

$$T_n = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & ... & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & ... & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & ... & 0 \\ . & . & . & & & \\ 0 & 0 & ... & ... & ... & \beta_{n-1} \\ 0 & ... & ... & 0 & \beta_{n-1} & \alpha_n \end{pmatrix}$$

Show that if a symmetric, real matrix A has a multiple eigenvalue, then the algorithm will terminate prematurely.
For this problem, we will show that if A has a multiple eigenvalue, then the Krylov subspace $\mathcal{K}_n$ will be dimension strictly less than n. To see this, note that, if A has a multiple eigenvalue, its minimal polynomial, μ_A will be degree strictly less than n (where A is $n \times n$). This means $\deg(\mu_A) \leq n - 1$. Consider the Krylov subspace $\mathcal{K}_n(b)$. Note that, because μ_A has highest power $n - 1$ at most, we have

$$\mu_A(A)b \in \mathcal{K}_n(b)$$

But $\mu_A(A) = 0$, so we have found a vector which is a linear combination of $A^i b$ for $0 \leq i \leq n - 1$ which is 0. Thus the vectors $b, Ab, ..., A^{n-1}b$ are not linearly independent, so $\mathcal{K}_n(b)$ must have dimension less than n.
Because the Krylov subspace is dimension-deficient, we know that at some point, the Gram-Schmit process for finding an orthonormal basis for $\mathcal{K}_n$ will fail for an iteration before n. This would mean that $\beta_i = 0$ for some $i < n$ and the Lanczos algorithm will terminate, prematurely.

b

Premature termination of the algorithm does not necessarily mean we have a multiple eigenvalue. If our initial starting vector b is chosen badly enough, i.e. if it does not have a component in every eigenspace for A, then repeated iterations of A will not hit every eigenspace for A and $\mathcal{K}_n$ will be dimension deficient.
However, if we consider the case where the algorithm terminates for almost every choice of b (as being in a lower dimensional subspace is a measure-0 condition), then we do get the implication that A has a multiple eigenvalue. This would be because pre-mature termination for almost all b means that there is a polynomial of degree $< n$ that zeros out A. Thus, the minimal polynomial of A is degree less than n, so there must be repeated factors in the characteristic polynomial.

5

Let A be a positive definite symmetric $n \times n$ matrix and we seek a solution to

$$Ax = b$$

Let $\{z_1, ..., z_n\}$ be a set of A-orthogonal non-zero vectors. Given a starting point x_0, define the conjugate directions

$$w_k = \frac{\langle z_k, b - Ax_{k-1}\rangle}{\langle z_k, Az_k\rangle}, \qquad x_k = x_{k-1} + w_k z_k$$

Prove that $Ax_n = b$.
To prove this, first we show what happens when you take the product of x_k and Az_k.

$$\langle Az_k, x_k\rangle = \langle Az_k, x_{k-1}\rangle + \frac{\langle z_k, b - Ax_{k-1}\rangle}{\langle z_k, Az_k\rangle}\langle Az_k, z_k\rangle$$

canceling out terms and using linearity we get

$$\langle Az_k, x_{k-1}\rangle + \langle z_k, b\rangle - \langle Ax_{k-1}, z_k\rangle$$

As A is symmetric, we have

$$\langle Ax_{k-1}, z_k\rangle = \langle x_{k-1}, Az_k\rangle$$

Thus we are left with

$$\langle Az_k, x_k\rangle = \langle z_k, b\rangle$$

Now if we similarly consider

$$\langle Az_j, x_k\rangle = \langle Az_j, x_{k-1}\rangle + \frac{\langle z_k, b - Ax_{k-1}\rangle}{\langle z_k, Az_k\rangle}\langle Az_j, z_k\rangle$$

for $j \neq k$ we have the second term is 0 so we are left with just $\langle Az_j, x_{k-1}\rangle$. We then get, by repeatedly applying our two formulas to x_n that

$$\langle Az_k, x_n\rangle = \langle z_k, b\rangle$$

This happens because multiplication by Az_k will reduce the index of x from n until it his k, leaving us with $\langle Az_k, b\rangle$.
Now consider that because all the z_k are A orthogonal, they are linearly independent, meaning they form a basis. Similarly, Az_i form a basis, because A is positive-definite. Let

$$b = \beta_1 Az_1 + ... + \beta_n Az_n$$

and

$$x_n = \alpha_1 z_1 + ... + \alpha_n z_n$$

We then have

$$\langle Az_k, x_n\rangle = \langle Az_k, \alpha_1 z_1 + ... + \alpha_n z_n\rangle = \alpha_k\langle Az_k, z_k\rangle$$

and also

$$\langle z_k, b\rangle = \langle z_k, \beta_1 Az_1 + ... + \beta_n Az_n\rangle = \beta_k\langle Az_k, z_k\rangle$$

So by our equality above, we have

$$\alpha_k = \beta_k$$

for all k. Thus,

$$Ax_n = \alpha_1 Az_1 + ... + \alpha_n Az_n = \beta_1 Az_1 + ... + \beta_n Az_n = b$$

6

Let $A = QR$ be a reduced QR factorization for a tall matrix A, which is $N \times n$ (for $N > n$). Prove that if R has m nonzero values on its diagonal, then $\text{rank}(A) \geq m$.
To prove this, we will show that $Ae_1, ..., Ae_m$ are linearly independent, and thus $\text{im}(A)$ has dimension at least m. Without loss of generality, suppose that the matrix R is ordered so the m nonzero entries are first. Then we have

$$Ae_1 = QRe_1 = Qr_{11}e_1 = r_{11}q_1$$

where q_i is the ith column of Q. Similarly, we have

$$Ae_2 = r_{21}q_1 + r_{22}q_2 \qquad Ae_3 = r_{31}q_1 + r_{32}q_2 + r_{33}q_3$$

In general we have $Ae_k = \sum_{i=1}^{k} r_{i,k}q_i$. Now suppose that for some α_is we have

$$\alpha_1 Ae_1 + ... + \alpha_m Ae_m = 0$$

We then consider

$$0 = \langle q_m, \alpha_1 Ae_1 + ... + \alpha_m Ae_m \rangle = r_{mm}\alpha_m$$

because Ae_m is the only vector in the sum that has a component in the q_m direction. As we assumed $r_{ii} \neq 0$ for $i \leq m$ we get $\alpha_m = 0$. We then get, by the same argument

$$0 = \langle q_{m-1}, \alpha_1 Ae_1 + ... + \alpha_m Ae_m \rangle = r_{(m-1),(m-1)}\alpha_{m-1}$$

so $\alpha_{m-1} = 0$. Continuing through all m coefficients we get that α_i must be 0 for all i. Meaning that the vectors Ae_i for $i \leq m$ are linearly independent, by definition. So A will have rank at least m.

Alterative method We can alternatively get the result via Sylvester's rank inequality, which states

$$\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$$

I this case we can apply it like

$$\text{rank}(Q) + \text{rank}(R) - n \leq \text{rank}(QR) = \text{rank}(A)$$

Note that Q is orthogonal, so $\text{rank}(Q) = n$. Also $\text{rank}(R) = m$ as the nonzero values in the diagonal correspond to nonzero eigenvalues. Then we get

$$n + m - n \leq \text{rank}(A)$$

so

$$m \leq \text{rank}(A)$$

7

Consider the GMRES algorithm, which combines the Arnoldi algorithm with a least squares solver. Define $\mathcal{K}_n = \text{span}\{b, Ab, ..., A^{n-1}b\}$, the nth Krylov subspace for b. Suppose that at iteration n we have "arnoldi breakdown" so $h_{n+1,n} = 0$.

a

Show that $A\mathcal{K}_n \subseteq \mathcal{K}_n$.
This problem is asking us to consider the case where Arnoldi breakdown occurs, which happens when the Krylov subspace for $n+1$ is dimension less than $n+1$. This happens because

$$h_{n+1,n} = ||Aq_n - \sum_{j=1}^{n} h_{j,n}q_j|| = 0$$

where $h_{j,n} = \langle q_j, Aq_n \rangle$. This would imply that $Aq_n \in \text{span}\{q_1, ..., q_n\}$. In general, for $j < n$ the Arnoldi iteration gives us

$$Aq_j = \sum_{i=1}^{j+1} h_{i,j}q_i$$

Now consider Au where $u \in \mathcal{K}_n$. We can write

$$u = \alpha_1 q_1 + ... + \alpha_n q_n$$

so

$$Au = \alpha_1 A q_1 + ... + \alpha_n A q_n$$

Note that for $j < n$, we have $Aq_j \in \text{span}\{q_1, ..., q_n\} \subseteq \mathcal{K}_n$. Also for Aq_n we showed that $Aq_n \in \text{span}\{q_1, ..., q_n\} \subseteq \mathcal{K}_n$ as well. So $Au \in \mathcal{K}_n$. So $A\mathcal{K}_n \subseteq \mathcal{K}_n$.

b

Show that this guarentees $x \in \mathcal{K}_n$ with x solving $Ax = b$.
In the case described above, we have that the algoritm breaks down at iteration n and no earlier, so $h_{j+1,j} \neq 0$ when $j < n$. This means that for all $j < n$ we get

$$Aq_j = \sum_{i=1}^{j+1} h_{i,j} q_i$$

Because A is invertible, the dimension of $A\mathcal{K}$ cannot be reduced from $\dim(\mathcal{K}_n)$. Thus $A\mathcal{K}_n$ is a $\dim(\mathcal{K}_n)$ dimensional subspace in $\mathcal{K}_n$. Thus $A\mathcal{K} = \mathcal{K}_n$. We then get that

$$b \in \mathcal{K}_n \in A\mathcal{K}_n$$

So, by definition of $A\mathcal{K}_n$, $b = Ax$ for some $x \in \mathcal{K}_n$.

c

Assuming A is diagonalizable and we are given $n < m$. Describe a method for determining b such that this breakdown will occur no later than step n.
We want a vector b such that $A\mathcal{K}_n(b) = \mathcal{K}_n(b)$. Consider the orthonormal eigenbasis for A with eigenvectors $v_1, ..., v_m$. If we choose $b \in \text{span}\{v_1, ..., v_n\}$ we get that

$$A^n b = \lambda_1^n v_1 + \lambda_2^n v_2 + ... + \lambda_n^n v_n \in \text{span}\{v_1, ..., v_n\}$$

So $A^j b \in \text{span}\{v_1, ..., v_n\}$ for all j. So the Krylov subspace must be dimension at most n. If $A\mathcal{K}_n$ was not contained in $\mathcal{K}_n$, then we would have $n+1$ linearly independent vectors in $\text{span}\{v_1, ..., v_n\}$, which is clearly a contradiction.

8

a

Consider the algorithm $x^{(k+1)} = x^{(k)} - \alpha_k M_k \nabla f(x^{(k)})$ where $f : \mathbb{R}^2 \to \mathbb{R}$ and $f \in C^1$ and

$$M_k = \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix}$$

and

$$\alpha_k = \text{argmin}_{\alpha \geq 0} f(x^{(k)} - \alpha M_k \nabla f(x^{(k)}))$$

At some iteration we have $\nabla f(x^{(k)}) = (1, -1)^T$. Find the largest range of values for a that guarantees $\alpha_k > 0$.
This is a modified version of the steepest descent algorithm, and we want to find the values for a such that $\alpha_k \neq 0$, meaning there will be a descent direction in $M_k \nabla f(x^{(k)})$. A descent direction will be a direction v in which $\nabla f(x)^T v > 0$. This is essentially asking to find the conditions in which

$$\nabla f(x^{(k)})^T M_k \nabla f(x^{(k)}) > 0$$

We can compute this value as

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -a \end{pmatrix} = a$$

so for the above quantity to be positive, we must have $a > 0$.

b

Consider a function $f : \mathbb{R}^d \to \mathbb{R}$ with $f(w) \geq c$ for all $w \in \mathbb{R}^d$. Assume there is some $L > 0$ such that

$$f(w') \leq f(w) + \nabla f(w)^T(w' - w) + \frac{L}{2}||w' - w||^2$$

for all $w, w' \in \mathbb{R}^d$. Show that there exists some $\alpha \in \mathbb{R}$ such that if we run gradient descent with fixed step size α, we get

$$\min_{0 \leq t \leq T-1} ||\nabla f(w^{(t)})||^2 \leq \frac{2L}{T}|f(w^{(0)}) - c|$$

If we consider $w' = w^{(t+1)} = w - \alpha \nabla f(w^{(t)})$, we have, from the descent lemma

$$\begin{aligned} f(w') &\leq f(w) - \nabla f(w)^T(\alpha \nabla f(w)) + \frac{L}{2}||\alpha \nabla f(w)||^2 \\ &= f(w) - ||\nabla f(w)||^2(\alpha - \frac{L\alpha^2}{2}) \end{aligned}$$

Rearranging we get

$$f(w') - f(w) \leq -||\nabla f(w)||^2 \alpha(1 - \frac{L\alpha}{2})$$

and

$$f(w) - f(w') \geq ||\nabla f(w)||^2 \alpha(1 - \frac{L\alpha}{2})$$

If we pick $\alpha = 2/L$ we have $\alpha(1 - \frac{L\alpha}{2}) = 1/(2L)$, so we get that

$$f(w) - f(w') \geq ||\nabla f(w)||^2 \frac{1}{2L}$$

Subbing in what w and w' are we have

$$f(x^{(k+1)}) - f(x^{(k)}) \geq ||\nabla f(x^{(k)})||^2 \frac{1}{2L}$$

We sum up the iterates of this inequality for the iterations 1 through T. Giving us

$$f(x^{(T)}) - f(x^{(0)}) \geq \sum_{j=0}^{T-1} ||\nabla f(x^{(j)})||^2 \frac{1}{2L} \geq \min_{0 \leq j \leq T-1} ||\nabla f(x^{(j)})||^2 \frac{T}{2L}$$

Then we use the fact that $f(w) \geq c$ for all w to get

$$c - f(x^{(0)}) \geq \min_{0 \leq j \leq T-1} ||\nabla f(x^{(j)})||^2 \frac{T}{2L}$$

and thus

$$|c - f(x^{(0)})| \frac{2L}{T} \geq \min_{0 \leq j \leq T-1} ||\nabla f(x^{(j)})||^2$$

Spring 2023

1

Consider $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$$

with $a \in \mathbb{C}$. Derive the condition on a for Gauss-Seidel to converge.
First note that Gauss-Seidel iterations take the form

$$x^{(k+1)} = Gx^{(k)} + M^{-1}b$$

where $M = L + D$ and $N = -U$, and $G = M^{-1}N$, where L, U, D denote the strictly upper, lower, and diagonal parts of the matrix. Then if we subtract x^* (the solution to $Ax^* = b$ from both sides, we get

$$\begin{aligned} x^{(k+1)} - x^* &= M^{-1}Nx^{(k)} + M^{-1}b - x^* \\ &= M^{-1}(M - A)x^{(k)} + M^{-1}Ax^* - x^* \\ &= (I - M^{-1}A)x^{(k)} - (I - M^{-1}A)x^* \\ &= (I - M^{-1}A)(x^{(k)} - x^*) \\ &= G(x^{(k)} - x^*) \end{aligned}$$

So we get that the error $x^{(k)} - x^*$ will go to zero if $\rho(G) < 1$, i.e. the largest eigenvalue of G (in absolute value) is less than 1. For this specified A we can compute

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$$

which has inverse

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{pmatrix}$$

so

$$G = M^{-1}N = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & -a^2 \end{pmatrix}$$

which has eigenvalues $\lambda = 0, -a^2$. So if $|a^2| \leq 1$ we get that the Gauss-Seidel iterations will converge.

2

Let $B \in \mathbb{R}^{n \times m}$ with $\text{rank}(B) = p$. Let $A = I - BB^T$ and we want to solve $Ax = b$ with the conjugate-gradient method. Assuming a solution exists in at most how many iterations would convergence happen?

3

The Lanczos iteration trigiagonalizes a Hermitian A. Assuming exact arithmetic, prove that q_j is orthogonal to $q_1, ..., q_{j-1}$.
Note that

$$\beta_{j-1}q_j = Aq_{j-1} - \beta_{j-2}q_{j-2} - \langle q_{j-1}, Aq_{j-1}\rangle q_{j-1}$$

The first and third term ensure that $\beta_j q_j \perp q_{j-1}$. Now, to proceed by induction, we assume that $q_{j-1}, ..., q_1$ are orthogonal. Then compute

$$\langle \beta_{j-1}q_j, q_{j-2}\rangle = q_{j-2}^T Aq_{j-1} - \beta_{j-2} - 0$$

Note that $q_{j-2}^T Aq_{j-1} = (Aq_{j-2})^T q_{j-1}$ because A is self-adjoint. And $Aq_{j-2} = \beta_{j-2}q_{j-1} + \beta_{j-3}q_{j-3} + \langle q_{j-2}, Aq_{j-2}\rangle q_{j-2}$. So

$$(Aq_{j-2})^T q_{j-1} = \langle \beta_{j-2}q_{j-1}, q_{j-1}\rangle = \beta_{j-2}$$

So

$$\langle \beta_{j-1}q_j, q_{j-2}\rangle = \beta_{j-2} - \beta_{j-2} = 0$$

Also note that for any other q_l we can write

$$\langle \beta_{j-1}q_j, q_l\rangle = (Aq_l)^T q_{j-1} + 0$$

Note that unless $l = j-2, j-1$ Aq_l will be a linear combination of q_p with $p < j-1$. So the inner product above will be 0. Thus we get that q_j is orthogonal to all q_i for $i < j$.

4

Let $A^* = -A$ (skew Hermitian). Prove $(I-A)^{-1}(I+A)$ is unitary.
To prove this, first note that

$$(I-A)^{-1}(I+A)^{-1} = ((I+A)(I-A))^{-1} = ((I-A)(I+A))^{-1} = (I+A)^{-1}(I-A)^{-1}$$

Also note that for any matrix $(B^*)^{-1} = (B^{-1})^*$. Then we multiply the matrix by its Hermitian conjugate to get

$$\begin{aligned}\left((I-A)^{-1}(I+A)\right)^* \left((I-A)^{-1}(I+A)\right) &= (I-A)\left((I-A)^*\right)^{-1}(I-A)^{-1}(I+A)\\ &= (I-A)(I+A)^{-1}(I-A)^{-1}(I+A)\\ &= (I-A)(I-A)^{-1}(I+A)^{-1}(I+A)\\ &= I^2 = I\end{aligned}$$

So the Hermitian conjugate of the matrix is its inverse, meaning the matrix is unitary.

5

Suppose $M = AB$ has a QR factorization. Prove or disprove the following

- The matrix A has a QR factorization
- The matrix B has a QR factorization

In general, a matrix A will have a QR factorization if it is tall, i.e. it is $m \times n$ with $m \geq n$.
This question is rather weird in its form as written. It slightly depends on your definition of the QR factorization. If we take the typical definition that $A = QR$ where Q has orthogonal columns and R is upper triangular then there are situations in which A has a QR factorization and B does not and visa versa. For example, let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

which has a QR factorization. But note that A has no QR factorization (B does however). Alternatively consider

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{pmatrix}$$

which is tall, so it has a QR factorization. If B has a QR factorization then Q would need to be $1 \times k$ and R would be $k \times 2$. If $k > 1$, then Q could not be orthogonal (because any two scalars are always linearly dependent). So $k = 1$ and $R = \begin{pmatrix} 2 & 1 \end{pmatrix}$

6

Let A be a square diagonalizable matrix with eigenvalues $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq \ldots.$ Consider the power method to compute an eigenvector q_1 of λ_1 which iterates as $z_{n+1} = Az_n/||Az_n||$.

a

Show (by example or in words) that if $|\lambda_1| = |\lambda_2|$, the power method does not necessarily converge to an eigenvector of λ_1.
If $|\lambda_1| = |\lambda_2|$, then we can have a part of the matrix that will not change the norm of a vector, but will simple rotate the vector of repeatedly swap between 2 different eigenvectors in a cyclic pattern. The prototypical example of this is the roation matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that this matrix is diagonalizable, and decomposes as

$$A = \begin{pmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

So $|\lambda_1| = |\lambda_2|$. Also note that if $||u|| = 1$, then $||Au|| = 1$. So the iteration will simple by $z_{n+1} = Az_n$. But this matrix simple rotates the vector z_n by $\pi/2$ about the origin, so clearly the iteration will never converge.

b

Prove that if $\lambda_1 = \lambda_2$, and $|\lambda_3| < |\lambda_1$, then the method still converges to an eigenvector of λ_1.

To prove this, we start by writing z_0 as a linear combination of eigenvectors for A, which is possible because A is diagonalizable. So

$$z_0 = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$$

where v_n are the eigenvectors for A for eigenvalues $\lambda_1, \lambda_2, ...$ Then note that

$$Az_0 = \alpha_1 \lambda_1 v_1 + ... + \alpha_n \lambda_n v_n$$

and then

$$z_1 = \frac{Az_0}{||Az_0||} = \frac{\alpha_1 \lambda_1 v_1 + ... + \alpha_n \lambda_n v_n}{\sqrt{\langle \alpha_1 \lambda_1 v_1 + ... + \alpha_n \lambda_n v_n, \alpha_1 \lambda_1 v_1 + ... + \alpha_n \lambda_n v_n \rangle}}$$

In the square root on the denominator, we can write the expression as

$$\sqrt{\lambda_1^2 \langle \alpha_1 v_1 + \alpha_2 v_2, \alpha_1 v_1 + \alpha_2 v_2 \rangle + \mathcal{O}(\lambda_1^{2-1})}$$

because all other eigenvalues are bounded by $\lambda_1 = \lambda_2$. Then note that

$$z_k = \frac{A^k z_0}{||Az_0|| * ||Az_1|| * ... * ||Az_k||} = \frac{\alpha_1 \lambda_1^k v_1 + ... + \alpha_n \lambda_n^k v_n}{\sqrt{\gamma_k}}$$

Note that $z_1 = Az_0/||Az_0||$ so

$$||Az_0|| * ||Az_1|| = \frac{||Az_0|| * ||A^2 z_0||}{||Az_0||} = ||A^2 z_0||$$

Similarly $||Az_0|| * ||Az_1|| * ... * ||Az_k|| = ||A^k z_0||$ So $\gamma_k = ||A^k z_0||^2$ will be a large polynomial in λ_1 with the first factor being

$$\langle \alpha_1 \lambda_1^k v_1 + ... + \alpha_n \lambda_n^k v_n, \alpha_1 \lambda_1^k v_1 + ... + \alpha_n \lambda_n^k v_n \rangle = \lambda_1^{2k} ||\alpha_1 v_1 + \alpha_2 v_2||^2 + \mathcal{O}(\lambda_1^k)$$

Then multiply the top and bottom by $1/\lambda_1^k$ and note that $\lambda_1 = \lambda_2$ to get

$$z_k = \frac{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 \left(\frac{\lambda_3}{\lambda_1}\right)^k v_3 + ... + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n}{\sqrt{||\alpha_1 v_1 + \alpha_2 v_2||^2 + \mathcal{O}(\frac{1}{\lambda_1^k})}}$$

Then we take the limit as $k \to \infty$ to get

$$z_k \to \frac{\alpha_1 v_1 + \alpha_2 v_2}{||\alpha_1 v_1 + \alpha_2 v_2||}$$

which is an eigenvector of A with eigenvalue λ_1.

7

I wrote this problem down wrong and then solved it completely for the wrong $f(x_1, x_2)$ so this is not actually the answer to the qual problem. I did not finish it for this reason. I also messed up and said that $\mu_1 < 0$ at some point which should not be possible. Consider the problem to find the extremizer of

$$x_1^2 + x_1 x_2$$

subject to $x_2^3 \leq x_1 \leq 2$.

a

Write down the KKT conditions for this problem and give all the points that satisfy them. The KKT conditions are that there exist $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ such that

$$\nabla f(x^*) + \lambda^T Dh(x^*) + \mu^T Dg(x^*) = 0$$

with $\mu^T g = 0$ and $\mu \geq 0$. Here h is the vector of equality constraints and g is the vector of inequality constraints ($g(x_1, x_2) \leq 0$). Here we just have g with

$$g = \left(\begin{array}{c} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{array} \right) = \left(\begin{array}{c} x_2^3 - x_1 \\ x_1 - 2 \end{array} \right) \leq 0$$

Then we have

$$Dg(x_1, x_2) = \begin{pmatrix} -1 & 3x_2^2 \\ 1 & 0 \end{pmatrix}$$

Then

$$\nabla f + \mu^T Dg = \begin{pmatrix} 2x_1 + x2 \\ x_1 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 3x_2^2 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 - \mu_1 + \mu_2 \\ x_1 + 3\mu_1 x_2^2 \end{pmatrix} = 0$$

So we get 4 equations

$$2x_1 + x_2 - \mu_1 + \mu_2 = 0$$
$$x_1 + 3\mu_1 x_2^2 = 0$$
$$\mu_1(x_2^3 - x_1) + \mu_2(x_1 - 2) = 0$$

(Note that the last equation actually gives two different equation as each term must be positive. First consider the case when $\mu_1 = \mu_2 = 0$. Then $2x_1 = -x_2$ and $x_1 = 0$. So $x_2 = 0$. This point satisfies $g(x_1, x_2) \leq 0$ because $g(x_1, x_2) = (0, -2)^T$. So the first point that satisfies the KKT conditions is

$$(0, 0, 0, 0)$$

Next let $\mu_1 = 0$, $\mu_2 \neq 0$. We then get

$$2x_1 + x_2 + \mu_2 = 0$$

and $x_1 = 0$. Also $\mu_2(x_1 - 2) = 0$ so $x_1 = 2$. This cannot be solved so there are no solutions where $\mu_1 = 0$ and $\mu_2 \neq 0$. Next, consider when $\mu_1 \neq 0$ and $\mu_2 = 0$. We get $2x_1 + x_2 - \mu_1 = 0$, and $x_1 = x_2^3$. If $x_2 = 0$, we get the solution we already obtained. If $x_2 \neq 0$ we get the second equation as

$$x_2^3 + 3\mu_1 x_2^2 = 0 \implies x_2 + 3\mu_1 = 0$$

So

$$2x_1 + x_2 - \mu_1 = 2x_2^3 + x_2 - x_2/3 = 0$$

We factor out x_2 again to get

$$2x_2^2 + \frac{2}{3} = 0$$

So $x_2 = 1/\sqrt{3}$. Then $x_1 = \frac{1}{3\sqrt{3}}$, which is certainly less than 2, so this point is valid. We then get that $\mu_1 = \frac{-1}{3\sqrt{3}}$. So the point satisfying the KKT conditions is

$$(\frac{1}{3\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{3\sqrt{3}}, 0)$$

Finally, consider when $\mu_1 \neq 0$ and $\mu_2 \neq 0$. We then have $x_1 = 2$ and $x_1 = x_2^3$. So $x_2 = \sqrt[3]{2}$. Then

$$2 + 3\mu_1 2^{2/3} = 0$$

So $\mu_1 = -2^{-1/3}/3$. And then $\mu_2 = -2x_1 - x_2 + \mu_1 = -4 - 2^{1/3} - \frac{1}{2^{1/3}3}$. Giving us our last point.

b

The second order necessary conditions state that

$$F + \mu_1 G_1 + \mu_2 G_2 = 0$$

is positive semi-definite on the tangent space $T(x) = \{y \in \mathbb{R}^2 : Dg_i y = 0\}$ for $i \in J(x)$, the active constraints of g, where F is the Hessian of f, G_1 is the Hessian of g_1 and G_2 is the Hessian of g_2. We can compute this as

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 & 0 \\ 0 & 6x_2 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 6\mu_1 x_2 \end{pmatrix}$$

I will do the analysis just for the second point. So $\mu_1 x_1 = -1/27$. In this case

$$L(x_1, x_2) = \begin{pmatrix} 2 & 1 \\ 1 & -2/9 \end{pmatrix}$$

In this case the only active constraint is g_1, so the tangent space will be all the vectors orthogonal to

$$Dg_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This will be vectors of the form $\begin{pmatrix} a \\ a \end{pmatrix}$. For this vector note that

$$v^T L v = \begin{pmatrix} a & a \end{pmatrix} \begin{pmatrix} 3a \\ 7/9a \end{pmatrix} = 3a^2 + \frac{7}{9}a^2 \geq 0$$

So this point satisfies the second order necessary conditions.

c

Determine whether the point satisfies the second order sufficient conditions. We will similarly just analyze the point above. For that point note that $v^T L v = a^2(34/9)$ so if $a \neq 0$ this will be strictly greater than 0. So this point satisfies the second order sufficient conditions. Technically, we need to check vectors orthogonal to active constraints where $\mu_i > 0$. But for g_1 we have $\mu_1 \neq 0$.

8

a

Consider a linear program in the standard form. Let x be a basic feasible solution. Show that if the reduced cost of every nonbasic variable is positive, then x is a unique solution.
A linear program is an optimization problem of the form $\min c^T x$ subject to $Ax = b$ and $x \geq 0$. Consider a basic feasible solution, $\overline{x}$. So $A\overline{x} = b$ and $\overline{x} \geq 0$. Then let the reduced cost of every nonbasic variable be positive. That is if

$$A = \begin{pmatrix} B & D \end{pmatrix}$$

where B is invertible, then $\overline{x} = \begin{pmatrix} x_b \\ x_d \end{pmatrix}$ and for every index j such that x_j is part of x_d we get the the reduced cost,

$$c_j - c^T B^{-1} a_j > 0$$

We can then define $\delta^j = e_j - B^{-1} a_j$ so the statement that the reduced cost is positive is simply $c^T \delta^j > 0$. We will show that every feasible direction (direction in which the equality constraint and inequality constraints are preserved, is a positive linear combination of δ^j).
Suppose v is a feasible direction. So for $\alpha > 0$ small enough, $\overline{x} + \alpha v$ satisfies the constraints. That is

$$A(\overline{x} + \alpha v) = A\overline{x} + \alpha A v = b$$

But $A\overline{x} = b$ so we get that $Av = 0$. Now consider splitting $v = \begin{pmatrix} v_b \\ v_d \end{pmatrix}$ as we do with $\overline{x}$. This means that

$$Bv_b = -Dv_d$$

So v_b is completely determined by v_d. If $v_d = e_j$ we get $v_b = -B^{-1}De_j = -B^{-1}a_j$. So that means

$$v_d = \sum_{j=1}^{k} \gamma_j e_j \implies v = \sum_{j=1}^{k} \gamma_j \delta^j$$

as $\delta^j = e_j - B^{-1}a_j$. Also note that we need $x + v \geq 0$ so v_d must only be a **positive** linear combination of e_j and thus v a positive linear combination of δ^j. So that means that, for any feasible direction,

$$c^T(x+v) = c^T x + \sum_{j=1}^{k} \gamma_j c^T \delta^j > c^T x$$

so long as all the γ_js are not 0. Meaning that x is a local minimizer of the objective. However, a linear program is convex so a local minimum is a global minimum. We get uniqueness because of the strict inequalities above.

b

Use duality to prove that the problem of minimizing $c^T x$ subject to $x \geq 0$ will have a solution if and only if $c \geq 0$. Also show that if a solution exists, then 0 is a solution.
For a problem of the for $\min c^T x$ subject to $Ax \geq b$ and $x \geq 0$ the dual problem is $\max b^T \lambda$ subject to $\lambda^T A \leq c^T$ and $\lambda \geq 0$. If the dual problem is bounded and has a nonempty feasible set, then the primal problem is nonempty and bounded. To see this note that a value in the feasible set for the dual, $\lambda A \geq c^T$ so $\lambda Ax \leq c^T x$ and also $\lambda^T b \leq \lambda^T Ax$. So $\lambda^T b \leq c^T x$. So if there is any feasible λ, we get that $c^T x$ is bounded (from below). Also if $c^T x$ is unbounded (from below), there is no λ that can be feasible for the dual problem. Additionally, note that if $b^T \lambda$ is unbounded, it means there can be no feasible x.
So if the dual problem has a solution (no-empty feasible set and bounded objective), then the primal will have a solution. I do not actually know how to get the implication in the other direction using duality. But you can prove the other direction using non-duality methods, as I do below.
Note that the problem above seeks to minimize $c^T x$ subject to $x \geq 0$. This can be thought of as the standard linear problem of minimizing $c^T x$ subject to $Ax \geq b$ with $A = 0$ and $b = 0$. The dual problem will then be to maximize 0λ subject to $0 \leq c^T$. If $c^T \not\geq 0$, then the dual problem will have no feasible solutions. If $c^T \geq 0$, then the dual will have a feasible solution (namely, any vector), and thus the primal problem will be bounded. The primal problem necessarily has a nonempty feasible set, so it will have a solution.
If $c^T \not\geq 0$, then let $c_i < 0$. The vector $x = \alpha e_i$ for any $\alpha > 0$ will have that $c^T \alpha e_i = \alpha c_i = -\alpha |c_i|$, which will grow to ∞ as $\alpha \to \infty$ (which will always remain in the feasible set).

9

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function with $f^* = \inf_x f(x) > -\infty$. Consider the subgradient method

$$x^{(k+1)} = x^{(k)} - \alpha_k g^k$$

where $g^k \in \partial f(x^{(k)})$. Show that if $0 < \alpha_k < 2\frac{f(x^{(k)}) - f^*}{||g^{(k)}||^2}$ then

$$||x^{(k+1)} - x^*|| < ||x^{(k)} - x^*||$$

for any optimal point x^*.
To prove this subtract x^* from both sides of our update rule to get

$$x^{(k+1)} - x^* = x^{(k)} - x^* - \alpha_k g^k$$

Then taking the norm of both sides via the inner product we get

$$\begin{aligned}||x^{(k+1)} - x^*||^2 &= \langle x^{(k+1)} - x^*, x^{(k+1)} - x^* \rangle \\ &= \langle x^{(k)} - x^* - \alpha_k g^k, x^{(k)} - x^* - \alpha_k g^k \rangle \\ &= ||x^{(k)} - x^*||^2 - 2\alpha_k \langle g^k, x^{(k)} - x^* \rangle + \alpha_k^2 ||g^k||^2\end{aligned}$$

Now use the fact that $g^k \in \partial f(x^{(k)})$ to say $f(x^*) - f(x^{(k)}) \geq \langle g^k, x^* - x^{(k)} \rangle$ so the above quantity becomes

$$\begin{aligned}&= ||x^{(k)} - x^*||^2 + 2\alpha_k \langle g^k, x^* - x^{(k)} \rangle + \alpha_k^2 ||g^k||^2 \\ &\leq ||x^{(k)} - x^*||^2 + 2\alpha_k (f(x^*) - f(x^{(k)})) + \alpha_k^2 ||g^k||^2\end{aligned}$$

Now use the bound on α_k to get that

$$\begin{aligned}&< ||x^{(k)} - x^*||^2 + 4\frac{(f(x^{(k)}) - f(x^*))(f(x^*) - f(x^{(k)}))}{||g^k||^2} + 4\frac{(f(x^{(k)}) - f(x^*))^2}{||g^k||^2} \\ &= ||x^{(k)} - x^*||^2 + 4\frac{(f(x^{(k)}) - f(x^*))^2}{||g^k||^2} - 4\frac{(f(x^{(k)}) - f(x^*))^2}{||g^k||^2} \\ &= ||x^{(k)} - x^*||^2\end{aligned}$$

So we get $||x^{(k+1)} - x^*||^2 < ||x^{(k)} - x^*||^2$. Meaning $||x^{(k+1)} - x^*|| < ||x^{(k)} - x^*||$.

Fall 2024

1

Let A be a unitary matrix.

a

Prove that the condition number of A is 1.

For this proof we can proceed in two different ways. Firstly, consider the SVD of A given by $U\Sigma V^*$. To get the values on the diagonal of Σ we consider the eigendecomposition of A^*A. But A is unitary, so $A^*A = I$. So all the eigenvalues of A^*A are 1. The singular values σ_i will then be the square roots of the eigenvalues of A^*A, so they will be 1. Then note that

$$\kappa(A) = \frac{\sigma_{max}}{\sigma_{min}} = 1$$

The second way to approach this proof is note that the singular values of A will be the absolute value of the eigenvalues of A, when A is diagonalizable. As a unitary matrix, the eigenvalues of A will all lie on the unit circle, and in particular have aboslute value 1. So all the singular values must be 1.

b

Prove that A is orthogonally diagonalizable.

This problem is trivial by invoking the spectral theorem (we have $A^*A = AA^* = I$, so A is normal and thus orthogonally diagonalizable), so we will prove the special case of the spectral theorem for unitary A.

To do this, we will first show that for any eigenspace E of A. We have

$$V = E \oplus E^\perp$$

and $E^\perp$ is A-invarient. Note that the direct product is obvious from the definition of $E^\perp$. The more interesting part is showing that $AE^\perp = E^\perp$. Consider some $v \in E^\perp$. So $\langle v, u \rangle = 0$ for all $u \in E$. Then consider

$$\langle Av, u \rangle = \langle v, A^*u \rangle$$

Because A and A^* commute, they perserve each other eigenspaces (this can be seem because if $Au = \lambda u$, then $AA^*u = A^*Au = \lambda A^*u$). So $A^*u \in E$ and thus $\langle v, A^*u \rangle = 0$. So $Av \in E^\perp$.

Now we proceed by induction on the dimension of the domain of A. The base case is easy, there is only one eigenvector. For the inductive step, note that, by the fundamental theorem of algebra, A must have an eigenvector in the space. Then A will have some eigenspace E and we can decompose

$$V = E + E^{\perp}$$

Then the operator $A|_{E^{\perp}}$ is still linear and by the inductive hypothesis, we get that it will be orthognally diagonalizable. So we have decomposed V into E along with the orthogonal eigenspaces we get from $A|_{E^{\perp}}$. So A will be othogonally diagonalizable.

2

Let A be a real square matrix with eigendecomposition $A = V\Lambda V^{-1}$. Suppose that the perturbation $A + \delta A$ has eigenvalue μ. Prove that there exists some eigenvalue λ of A such that $|\lambda - \mu| \leq \kappa(V)||\delta A||$ (where $\kappa(A)$ is the condition number of V and $||\cdot||$ is the spectral norm).

For this problem we will first prove the hint: if μ is not an eigenvalue of A then -1 is an eigenvalue of $(\Lambda - \mu I)^{-1}V^{-1}\delta AV$. To see this suppose w is an eigenvalue of $A + \delta A$ with eigenvalue μ. Then

$$(A + \delta A)w = \mu w$$

Rearranging this expression gives us

$$(A - I\mu)w = -\delta Aw$$

Now define $w^1 = V^{-1}w$. We can then write

$$V(\Lambda - \mu I)w^1 = V(\Lambda - \mu I)V^{-1}w = (A - I\mu)w = -\delta Aw = -\delta AVw^1$$

We can multiply both expressions by V^{-1} then to get

$$(\Lambda - \mu I)w^1 = -V^{-1}\delta AVw^1$$

And finally, note that $(\Lambda - \mu I)^{-1}$ will be nonsingular because μ is not an eigenvalue of A, we get

$$w^1 = (\Lambda - \mu I)^{-1}V^{-1}\delta AV$$

Next we show that the spectral norm is submultiplicative. That is $||AB|| \leq ||A||||B||$. To see this, consider the SVD of A, B so

$$AB = U\Sigma V^*\hat{U}\hat{\Sigma}\hat{V}^*$$

Because $U, V, \hat{U}, \hat{V}$ can never change the magnitude of a vector, the spectral norm will depend only on the product $\Sigma\hat{\Sigma}$. As both matices are diagonal, the product will just be the element-wise product of the diagonals, will will certainly be less than $\sigma_{max}\hat{\sigma_{max}}$. So the spectral norm is sub-multiplicative.
Now, for the problem, if μ is an eigenvalue of A, then the inequality is true trivially. So suppose not. Then we have that -1 is an eigenvalue of $(\Lambda - \mu I)^{-1}V^{-1}\delta AV$. Thus, the spetral norm will satisify

$$1 \leq ||(\Lambda - \mu I)^{-1}V^{-1}\delta AV|| \leq ||(\Lambda - \mu I)^{-1}||||\delta A||(V)$$

Note that $||\Lambda - \mu I|| = \frac{1}{\lambda_j - \mu}$ for some λ_j because the matrix is diagonal so we know all its eigenvalues. Thus, for some λ_j we get

$$(\lambda_j - \mu) \leq \kappa(V)||\delta A||$$

As desired.